Dr. Simkin's example 2, the linear oscillator, was demonstrated in Ref. 2 without the necessity of Lagrange multipliers and equations of constraint. The direct analytical solution to a nonlinear oscillator is presented in Ref. 15. Example 3, of course, may be formulated and the direct analytical solution obtained through Hamilton's law without use of Dirac functions and without use of Lagrange multipliers.

It is significant that all three of Dr. Simkins' examples treat conservative systems. Direct analytical solutions to both conservative and nonconservative systems, both discrete[2,8,15] and for continuua[3,9,14,24] have now been demonstrated through application of Hamilton's law. With this law, it makes little difference whether the system is conservative or nonconservative (both conservativeness and stationarity were *assumed* by Lagrange in order to produce his rigorous proof of the principle of least action).

I must again express my appreciation to Dr. Simkins for his reference to my papers. However, in my opinion, his generation of "unconstrained variational statements" fails on at least two counts, simplicity and generality.

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Reply by Author to C. D. Bailey

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A BRIEF review may help to clarify the relationship of my own work[1] to that of Professor Bailey.[2] Hamilton's principle may be derived in a few steps from D'Alembert's principle. The next to last step in this derivation yields the variational statement[3]:

$$\int_{t_1}^{t_2} (\delta T + \delta \bar{W})\,dt - \left[\sum_{i=1}^{N} m_i \dot{q}_i \delta q_i\right]_{t_1}^{t_2} = 0 \tag{1}$$

T is the kinetic energy and $\delta \bar{W}$ is the virtual work done by the impressed forces on a system of N particles having mass m_i and coordinates q_i. The bar indicates that, in general, all of the terms of $\delta \bar{W}$ are not derivable from a scalar function. Now Eq. (1) is a valid physical statement that can be employed, as Bailey has done, as a basis for achieving approximate solutions to dynamics problems. For these purposes one need not proceed any further into the derivation.

Hamilton, of course, went on to require that $\delta q_i (t_1) = \delta q_i (t_2) = 0$ and $\delta \bar{W} = \delta W = -\delta V$; i.e., the virtual work term is such that it can be expressed as the variation of a single scalar function. The result is Hamilton's *principle*:

$$\delta \int_{t_1}^{t_2} (T - V)\,dt = 0 \tag{2}$$

One notes in passing that Lanczos[4] has shown that even when V is explicitly time variant, Eq. (2) still applies. Thus Hamilton's principle is valid for a limited class of nonconservative systems.

In the event the second requirement is removed, the result is called Hamilton's *extended principle*[3]:

$$\int_{t_1}^{t_2} (\delta T + \delta \bar{W})\,dt = 0 \tag{3}$$

Note that neither Eq. (1) nor Eq. (3) is of the form $\delta I = 0$; hence neither is a "principle" in the same sense that Eq. (2) is—they might better be called *variational statements*. This does not, however, prevent either from being used as a basis for achieving approximate solutions to many problems in mechanics. In fact, Eq. (3) is the most widely used form of Hamilton's principle for such applications—Smith's comments notwithstanding.[5]

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The difficulty in using either Eqs. (2) or (3) for approximating solutions to initial value problems is that the $q_i(t_2)$ are unknown and hence the $\delta q_i(t_2)$ must not be required to vanish. On the other hand, the $q_i(t_1)$ and $\dot{q}_i(t_1)$ are specified, hence

$$\delta q_i(t_1) = \delta \dot{q}_i(t_1) = 0 \tag{4}$$

The vanishing of the variations of the initial values means that any approximation functions chosen to solve an initial value problem must identically satisfy the prescribed initial values. Bailey does this simply by employing mth-order polynomials like

$$q_i(t) = q_i(0) + \dot{q}_i(0)t + \sum_{j=2}^{M} A_j \left(\frac{t}{t_2}\right)^j \tag{5}$$

where t_1 has been taken to be zero. Thus, in using Eq. (1), the trial functions are *constrained* a priori to satisfy the initial conditions and Eq. (1) when subjected to constraint Eqs. (4) may thereby be called a *constrained variational statement*. (Similarly, Eqs. (2) and (3) are also constrained variational statements.)

The thought arises that the use of polynomial trial functions may not always be practical or converge the fastest. Other functions, however, may not be made to satisfy the constraining initial conditions as readily. It would therefore be useful if one could formulate a variational statement which does not require the use of trial functions that identically satisfy the specified initial conditions. That is, the requirement $\delta q_i(0) = \delta \dot{q}_i(0) = 0$ could be abandoned.

Now Tiersten[6] was aware that, when dealing with boundary value problems, the requirement (constraint) that approximating functions satisfy certain boundary conditions can be relaxed through the use of Lagrange multipliers, and he employed this technique in solving elasticity problems involving one or more surfaces of material discontinuity. Such problems could not be solved by conventional variational methods since nothing was known about the boundary conditions to be imposed at the interfaces. Tiersten also recognized that this was very analogous to the lack of specified conditions at t_2 when attempting to deal with initial value problems by a variational approach. He then proceeded to apply the method of Lagrange multipliers to Hamilton's extended principle in order to remove the constraints on the trial functions, thus freeing all variations of constraints in space and time. By this means the physics of the problem were preserved, that is, the end result remained a physical balance. (An arbitrary weighting of residuals was thus ruled out implicity in Tiersten's work.) To this end Tiersten was able to show that his resulting variational statement reproduced all of the equations of motion and boundary and initial conditions when the variations were performed.

While Tiersten demonstrated the use of his *unconstrained* variational statement for solving a boundary value problem, its success in treating an initial value problem was never tested. Furthermore, Tiersten did not make complete deductive use of the Lagrange multiplier method in deriving the results for the time domain. The purpose of my own work was to show that Tiersten's variational statement does indeed give good results to initial-boundary value problems and can be derived for a given boundary-initial value problem via a Galerkin procedure in which unknown multipliers are deducible from successive integrations by parts.

Now, a few final remarks as to Bailey's concerns:

1) If Eq. (10b) of Ref. 1 appears a bit abstruse, it is only because it is written for the full four-dimensional space-time continuum. For particle mechanics problems of the variety treated by Bailey, the simplification is considerable, e.g., the linear oscillator:

$$\int_0^{t_1} (\dot{u}\delta\dot{u} - \omega^2 u\delta u)\,dt - \dot{u}(t_1)\delta u(t_1) + v_0\delta u(0) + u(0)\delta\dot{u}(0) - u_0\delta\dot{u}(0) = 0 \tag{6}$$

This is hardly more complex than Bailey's statement[2] (Hamilton's Law of Varying Action) for the problem; i.e.:

$$0 = \int_0^t \left(q'\delta q' - \frac{Kt_1^2}{M} q\delta q\right) d\tau - q'(t)\delta q(t) + q'(0)\delta q(0) \tag{7}$$

Note the explicitness of the specified quantities u_0, v_0 in Eq. (6) and that Eq. (6) reduces to Eq. (7) if the initial variation, $\delta\dot{u}(0)$ vanishes.

2) Bailey states that I have treated only conservative systems. The third application of Ref. 1 treated a problem with time-variant mass. I doubt if anyone would consider this a conservative system.

3) I believe Ref. 1 presents a useful method for generating physically meaningful variational statements for the solution of a wide class of boundary-initial value problems given the differential equation and boundary and initial conditions. The method allows a more general choice of approximating functions than does Hamilton's Law of Varying Action [Eq. (1)] and in practice presents little, if any, additional complication.

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